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Bogomol'nyi bounds for gravitational cosmic strings

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ABSTRACT: We present a new method for finding lower bounds on the energy of topological cosmic string solutions in gravitational field theories. This new method produces bounds that are valid over the entire space of solutions, unlike the traditional approach, where the bounds obtained are only valid for cylindrically symmetric solutions. This method is shown to be a generalisation of the well-known Bogomol'nyi procedure for non-gravitational theories and as such, it can be used to find gravitational Bogomol'nyi bounds for models wherever the traditional Bogomol'nyi procedure can be applied in the non-gravitational limit. Furthermore, this method yields Bogomol'nyi equations that do not appear to rule out the existence of asymmetric bound-saturating solutions.

KEYWORDS: Solitons Monopoles and Instantons, Classical Theories of Gravity, Supergravity Models.

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1. Introduction

Topological defects are of considerable interest in many areas of theoretical and mathematical physics, and find application in topics as diverse as superconductivity, nuclear physics and cosmology, as well as having interesting mathematical properties in their own right. Whilst most of these fields are concerned with the study of topological defects in non-dynamical and often flat spacetimes, within the field of cosmology it becomes important to examine the effect of gravity on the properties and behaviour of these objects.

When studying topological defects, one is often interested in finding the static configurations that minimise the total energy within each topologically distinct class of boundary conditions. Such configurations are the stable, classical ground states of the theory, and also form a convenient basis for numerical and analytical studies of low energy defect dynamics.

In Minkowski space, or other non-dynamical, highly symmetrical spacetimes, there is an established method, attributed to Bogomol'nyi, for finding such minimum-energy field configurations in many models that admit solitonic solutions (for a review, see [1]). It involves making use of a clever rearrangement of the energy-momentum tensor to write the total energy as

$$E = \int d^3\mathbf{x} T_0^0 = |Q| + P, \quad (1.1)$$

where Q is a topologically conserved charge, related to the asymptotic boundary conditions, and P is a manifestly non-negative spatial volume integral. This leads to the following lower bound on the energy of defects

$$E \geq |Q|, \quad (1.2)$$

which is called a Bogomol'nyi bound. The energy is minimised when P is zero, and by finding the conditions under which P vanishes, we obtain a set of field equations, called Bogomol'nyi equations, that characterise the minimum-energy field configurations.

However, for gravitational field theories, such energy bounds have been harder to come by, due to the difficulty in finding a suitable expression for the total energy, and the more complicated form of the energy-momentum tensor for a general metric. One way forward is to reduce the number of degrees of freedom in the metric by imposing certain exact symmetries on the spacetime [2]. When working within a well-chosen class of highly symmetric metrics, the expressions for the total energy and the energy-momentum tensor become very similar to their counterparts from the corresponding non-gravitational theory, and therefore we can perform a similar rearrangement to minimise the energy.

This approach is widely used [3–5], and the energy bounds and first-order equations that are derived in this manner are usually called Bogomol’nyi bounds and equations. The beauty of this method is that it is a natural extension of the familiar non-gravitational Bogomol’nyi method – in fact, a gravitational bound of this sort follows wherever a similar bound exists for the corresponding non-gravitational theory. This result, which suggests that Bogomol’nyi bounds generally survive coupling to gravity [6], means that there is an implicit assumption in much of the literature on gravitational topological defects that a non-gravitational Bogomol’nyi bound is enough to establish the stability of bound-saturating solutions even after gravity is taken into account [7, 8].

However, as the assumption of symmetry is made *prior* to minimising the energy, we cannot in fact preclude the possibility that the energy bounds provided by this method may be saturated, or even violated, by defects that do not possess the assumed symmetries. Intuitively speaking, we do not expect such bound-violating solutions to exist — we would be surprised to find that non-gravitational Bogomol’nyi bounds do not survive coupling to gravity. Nevertheless, without a more rigorous derivation of Bogomol’nyi bounds for gravitational theories, the stability of the widely-studied defect solutions that saturate these bounds is called into question.

Furthermore, it is often the case in non-gravitational theories that there exist multi-defect solutions that saturate the Bogomol’nyi bounds, at least for some region of the parameter space. Again, it seems reasonable to consider whether such solutions survive the coupling to gravity — however, unless one can guess a sufficiently accurate ansatz for the metric beforehand, the Bogomol’nyi technique of [2] cannot help us answer this question.

For these reasons, it would be worthwhile to pursue an alternative method for finding minimum-energy solutions in gravitational field theories that does not depend on making prior assumptions of symmetry. The pursuit of such a method would involve tackling the problems mentioned above — that of finding an appropriate expression for the energy, and that of rearranging this expression in the presence of a large number of degrees of freedom in the metric — head on.

The reader may have already noticed a striking resemblance between the problem described here and the positive energy theorem in general relativity. In fact, Witten’s proof of the positive energy theorem [9], with its use of a spinorial expression for the total energy, has already proved rather useful in establishing Bogomol’nyi bounds for certain theories. Using techniques derived from this proof, full Bogomol’nyi bounds have been constructed for certain three-dimensional [10] and four-dimensional [11] supergravity models with D -term symmetry breaking.

In this paper, we extend these results and demonstrate how, using techniques from the positive energy theorem, we may derive Bogomol'nyi bounds for any gravitational field theory wherever a similar bound exists for its non-gravitational counterpart. Due to the cosmological motivation for this study, we only consider cosmic strings from now on. However, we expect the methods presented here to be applicable, following appropriate modifications, to defects of other dimensionalities, such as domain walls and monopoles.

The rest of this paper is organised as follows. In section 2 we describe the asymptotic structure of a spacetime containing a long cosmic string, and consider how to express the total energy of such a spacetime. We use this expression in section 3 to find a Bogomol'nyi bound for the gravitational version of the abelian-Higgs model and subsequently examine how this model forms the basis for finding Bogomol'nyi bounds for many other field theories. Then in section 4, we compare our gravitational Bogomol'nyi procedure to the traditional non-gravitational Bogomol'nyi procedure, and show that the former is really a generalisation of the latter. In this way we confirm that non-gravitational Bogomol'nyi bounds, and the single-vortex solutions that saturate them, do survive coupling to gravity. We conclude in section 5.

2. Cosmic string spacetimes

If we are to minimise the total energy of a spacetime without relying on working within a class of highly symmetric metrics, then we must first identify a suitable expression for the total energy. In general relativity, the notion of total energy is closely tied up with the asymptotic structure of the spacetime under consideration. This is because the energy is a global quantity, dependent on the behaviour of the fields at every point on some hypersurface that stretches to infinity. Therefore, without being able to effectively compactify the spacetime, by specifying an appropriate asymptotic structure, we cannot hope to calculate the total energy of a system.

Where we have a compact source, the usual definitions of the energy (such as the ADM energy) stem from the canonical notion of an asymptotically flat spacetime [12]. Such a spacetime can be compactified, with a single point representing spatial infinity, and the spacetime becomes asymptotically flat in every direction. However, it is clear that this standard notion of asymptotic flatness is not appropriate for describing a long cosmic string — essentially because there is now an axial direction (running parallel to the string) along which fields do not fall to zero and we do not reach asymptotic flatness.

To resolve this problem, we must modify our notion of asymptotic flatness for a cosmic string spacetime, by distinguishing between radial infinity (where we do have asymptotic flatness) and the asymptotic behaviour in the axial direction, on which we have to impose suitable conditions in order to have a well-defined energy. We shall accomplish this by compactifying one spatial dimension on a circle of circumference L_z and wrapping the cosmic string around this circle. In the limit $L_z \rightarrow \infty$, edge effects should vanish, and the results we obtain should reasonably represent the properties of an infinitely long string.

Having thus described the asymptotic structure of a cosmic string spacetime, we note that, due to asymptotic flatness, there exists a neighbourhood of radial infinity, the *asymptotic*

otic region, in which we can find asymptotic cylindrical coordinates, (t, r, θ, z) for r greater than some constant r_0 , in which the metric tends to the following limit as $r \rightarrow \infty$:

$$ds^2 = dt^2 - dr^2 - (1 - \delta/2\pi)^2 r^2 d\theta^2 - dz^2, \quad (2.1)$$

where δ is the conical deficit angle. These coordinates shall turn out to be useful later on, when we examine the behaviour of fields near radial infinity. In order to fix the deficit angle δ , we recall that the solutions we are interested in should asymptotically tend to the static, cylindrically symmetric bound-saturating solutions that have already been found using the traditional Bogomol'nyi method of [2] (as these are the solutions that are relevant to a discussion about the stability of static, cylindrically symmetric solutions). For the static cylindrically symmetric solutions, one finds that $\delta = 2\pi|Q|$, where Q is the topological charge of the string. Therefore, we fix δ in a similar manner here.

We now turn to the question of how to define the total energy of a cosmic string spacetime. In a *canonical* asymptotically flat spacetime, the ADM energy is defined with respect to some maximal spacelike hypersurface S (i.e. a spacelike hypersurface that extends to spatial infinity) in terms of a surface integral over the asymptotic boundary of S , ∂S , at spatial infinity. Given such an integral expression for the ADM energy, it seems reasonable to speculate that the energy of a cosmic string spacetime can be given by a similar expression, with the only difference being that we replace spatial infinity by radial infinity, resulting in ∂S having the topology of a torus, rather than a sphere. If this is the case (as is confirmed in appendix A), then in order to find an expression for the energy of a cosmic string spacetime that satisfies the requirements set out in the Introduction, we only need find an appropriate expression for the ADM energy satisfying the same conditions: we expect this expression to carry over to the cosmic string spacetime following a simple change of the asymptotic surface of integration.

All that now remains is to identify a suitable expression for the ADM energy — one that, as described in the Introduction, is likely to admit a Bogomol'nyi rearrangement for a fully general metric. In particular, it would be ideal if the energy expression had a clear connection to the Minkowski spacetime expression for the total energy, in terms of a volume integral over the energy-momentum tensor.

Such an energy expression has been provided by Nester [13] during his proof of the positive energy theorem:¹

$$p^\mu u_\mu^\infty = \frac{1}{2} \int_{\partial S} dS_{\mu\nu} E^{\mu\nu}, \quad (2.2)$$

where

$$E^{\mu\nu} = i\varepsilon^{\mu\nu\rho\sigma} (\bar{\eta}\gamma_5\gamma_\rho\nabla_\sigma\eta - \overline{\nabla_\sigma\eta}\gamma_5\gamma_\rho\eta) \quad (2.3)$$

is the Witten-Nester 2-form.² The parameter η is an arbitrary Dirac spinor field that is asymptotically Killing ($\nabla_\mu\eta \rightarrow 0$), and $u^\mu = \bar{\eta}\gamma^\mu\eta$ is hence an asymptotically constant, timelike vector field. The asymptotic 4-vector $u^{\infty\mu}$ is the limit of u^μ at spatial infinity, and

¹Throughout this paper we work in natural units, with $8\pi G = 1$.

²The spinor covariant derivative is given by $\nabla_\mu\eta = \partial_\mu\eta + \frac{1}{4}\omega_\mu^{\nu\rho}\gamma_{\underline{\nu\rho}}\eta$, where $\gamma_{\mu\nu} = \gamma_{[\underline{\mu}\gamma_{\underline{\nu}]}$. Underlined indices are used to represent frame (tetrad) components.

represents the 4-velocity of the observer at spatial infinity who is measuring the energy of the system.

Having converted this expression into a volume integral with the aid of the divergence theorem, we apply the identity

$$\nabla_{[\mu}\nabla_{\nu]}\eta = -\frac{1}{8}R^{\rho\sigma}{}_{\mu\nu}\gamma_{\rho\sigma}\eta, \tag{2.4}$$

and Einstein's equation to find that

$$p^\mu u_\mu^\infty = \int_S dS_\mu \left\{ T^\mu{}_\nu u^\nu - 2\overline{\nabla}_\nu\eta\gamma^{\nu\mu\rho}\nabla_\rho\eta \right\}. \tag{2.5}$$

This integral is very similar to the Minkowski spacetime expression for the energy. In fact, in a static spacetime, with S normal to the timelike and Killing t -direction, we can choose η to be an exact Killing spinor such that $u^\mu = (1, 0, 0, 0)$ everywhere, and the above expression reduces to

$$p^0 = \int_S dV T^0_0, \tag{2.6}$$

which is exactly the Minkowski spacetime expression for the energy – the very expression that is rearranged during the non-gravitational Bogomol'nyi procedure. This is an encouraging sign that we may be able to rearrange the Witten-Nester energy expression, in an analogous manner to the non-gravitational rearrangement of (2.6), in order to find Bogomol'nyi bounds in the presence of gravity.

3. Bogomol'nyi bounds for gravitational field theories

We shall now examine how the Witten-Nester energy expression may be used to find Bogomol'nyi bounds for gravitational field theories in cosmic string spacetimes.

Upon transferring the Witten-Nester energy expression to a cosmic string spacetime, we immediately encounter a hitch: there are no globally well-defined asymptotically Killing spinors in a cosmic string spacetime with non-zero deficit angle. This can be seen quite simply by noting that the θ component of the Killing spinor equation is asymptotically

$$\partial_\theta\eta - \frac{1}{2}C'\gamma_{12}\eta = 0, \tag{3.1}$$

where $C' = 1 - \delta/2\pi$. This has the general solution

$$\eta = \eta_+ e^{\frac{iC'\theta}{2}} + \eta_- e^{-\frac{iC'\theta}{2}}, \tag{3.2}$$

where η_\pm are coordinate-constant spinors satisfying the projection conditions

$$(1 \pm i\gamma_{12})\eta_\pm = 0. \tag{3.3}$$

Therefore, even if we set either η_+ or η_- to zero, we can only obtain a globally well-defined spinor η if $C' = 1$ (and hence $\delta = 0$).

In order to circumvent this problem, let us now suppose that it is possible to find some current, \mathcal{J}_μ , constructed from the matter fields, such that

$$\int_{S_\infty} \mathcal{J}_\mu dx^\mu = 2\pi Q, \tag{3.4}$$

where Q is the topological charge of the cosmic string and S_∞ is any closed curve at radial infinity that encircles the cosmic string once. Using this current we can define a *modified* covariant spinor derivative $\hat{\nabla}_\mu$ by including an extra connection term as follows:

$$\hat{\nabla}_\mu \eta = \nabla_\mu \eta + \frac{i}{2} \mathcal{J}_\mu \eta. \tag{3.5}$$

We may now consider whether there exist any spinors that asymptotically satisfy the *modified* Killing spinor equation $\hat{\nabla}_\mu \eta = 0$. In fact, we can solve this equation asymptotically in a similar manner to before, and now find the general asymptotic solution

$$\eta = \eta_+ e^{\frac{i(C'-Q)\theta}{2}} + \eta_- e^{\frac{-i(C'+Q)\theta}{2}}. \tag{3.6}$$

This solution is globally well-defined, provided that $\eta_{\text{sign}(Q)} = 0$.

We shall see that, for many models admitting solitonic string solutions, a current \mathcal{J}_μ , satisfying condition (3.4) does exist. For such models, we can therefore find asymptotically *modified*-Killing spinors, which satisfy the asymptotic projection condition

$$(1 + i\kappa\gamma_{12})\eta \rightarrow 0, \tag{3.7}$$

where $\kappa = \text{sign}(Q)$. We also note that this asymptotic projection condition implies that, given an asymptotically modified-Killing spinor η , we can always find an asymptotic cylindrical coordinate system in which the two asymptotically constant (and Killing) vectors $u^\mu = \bar{\eta}\gamma_\mu\eta$ and $v^\mu = \bar{\eta}\gamma_5\gamma_\mu\eta$ have the following limits as $r \rightarrow \infty$:

$$u_\mu^\infty dx^\mu = dt, \quad v_\mu^\infty dx^\mu = \kappa dz. \tag{3.8}$$

Before proceeding, we ought to eliminate a potential source for confusion. In supergravity theories, the notation $\hat{\nabla}_\mu$ is often used to denote a particular choice of modified spinor derivative — essentially one where the current \mathcal{J}_μ of our notation is identified with the gravitino U(1) connection A_μ^B . Although, as we shall see later on, such a choice allows us to define modified-Killing spinors for certain models, there are other models in which the holonomy of A_μ^B no longer leads to the cancellation in (3.6) that is required for the existence of modified-Killing spinors. Furthermore, we would like our Bogomol'nyi procedure to be just as applicable as its non-gravitational counterpart, which can be applied to a model without regard to any supersymmetric extension the model may or may not admit. For these reasons, we choose to define the modified spinor derivative more generally, so that we can make a more judicious choice of connection that allows for the existence of asymptotically modified-Killing spinors.

Using this modified spinor derivative, we may define a modified Witten-Nester 2-form $\hat{E}^{\mu\nu}$ in the following manner:

$$\hat{E}^{\mu\nu} = i\varepsilon^{\mu\nu\rho\sigma} \left(\bar{\eta}\gamma_5\gamma_\rho\hat{\nabla}_\sigma\eta - \overline{\hat{\nabla}_\sigma\eta}\gamma_5\gamma_\rho\eta \right). \tag{3.9}$$

Integrating $\hat{E}^{\mu\nu}$ over ∂S , we find that the inclusion of the \mathcal{J}_μ connection gives

$$\frac{1}{2} \int_{\partial S} dS_{\mu\nu} \hat{E}^{\mu\nu} = \frac{1}{2} \int_{\partial S} dS_{\mu\nu} E^{\mu\nu} - 2\pi\kappa Q L_z, \quad (3.10)$$

where $E^{\mu\nu}$ is the original Witten-Nester 2-form, defined as in (2.3).

As argued in appendix A, the integral of $E^{\mu\nu}$ gives the energy of the cosmic string spacetime. Therefore, dividing (3.10) by L_z , we find that the energy per unit length, μ , satisfies

$$\mu = 2\pi|Q| + \frac{1}{2L_z} \int_{\partial S} dS_{\mu\nu} \hat{E}^{\mu\nu}. \quad (3.11)$$

Hence we can establish the Bogomol'nyi bound

$$\mu \geq 2\pi|Q|, \quad (3.12)$$

if we can demonstrate that the integral of $\hat{E}^{\mu\nu}$ in (3.11) is non-negative. To this end, we repeat the manipulations that took us from the surface integral (2.2) to the volume integral (2.5), and find that

$$\frac{1}{2} \int_{\partial S} dS_{\mu\nu} \hat{E}^{\mu\nu} = \int_S dS_\mu \left\{ T^\mu{}_\nu u^\nu - \epsilon^{\mu\nu\rho\sigma} \partial_\nu \mathcal{J}_\rho v_\sigma + 2\overline{\hat{\nabla}}_\nu \eta \gamma^{\nu\mu\rho} \hat{\nabla}_\rho \eta \right\}. \quad (3.13)$$

If we choose the S to be normal to the 0-direction, this becomes

$$\frac{1}{2} \int_{\partial S} dS_{\mu\nu} \hat{E}^{\mu\nu} = \int_S dS_0 \left\{ T^0{}_\nu u^\nu - \epsilon^{0ijk} \partial_i \mathcal{J}_j v_k - 2g^{ij} \hat{\nabla}_i \eta^\dagger \hat{\nabla}_j \eta - 2(\gamma^i \hat{\nabla}_i \eta)^\dagger (\gamma^j \hat{\nabla}_j \eta) \right\}. \quad (3.14)$$

The third term in this integral is manifestly positive-definite, whilst the fourth term is negative-definite. However, the fourth term vanishes if the spinor parameter η satisfies

$$\gamma^i \hat{\nabla}_i \eta = 0 \quad (3.15)$$

throughout S . In fact, as we shall demonstrate shortly, we can always choose η to satisfy this condition, as long as the inequality

$$T^0{}_\nu u^\nu - \epsilon^{0ijk} \partial_i \mathcal{J}_j v_k \geq 0 \quad (3.16)$$

is satisfied throughout S . Therefore, the existence of a current \mathcal{J}_μ that satisfies the inequality (3.16) is all that is required to show that the integral (3.14) is non-negative.

To summarise, we have found that the Bogomol'nyi bound (3.12) can be established as long as we can find a current, \mathcal{J}_μ , satisfying the asymptotic property (3.4), such that the inequality (3.16) holds throughout S .

Let us now return to the condition (3.15). This is a modified version of the Witten-Nester condition, which was originally introduced by Witten during his proof of the positive energy theorem [9]. Adapting Witten's arguments, we shall now show that there always exists an asymptotically modified-Killing spinor field η that satisfies this condition throughout S .

We begin by defining a spinor field η_0 that, in the asymptotic region, takes the value

$$\eta_0 = \eta_\kappa e^{-\kappa \frac{i\theta}{2}}, \quad (3.17)$$

where η_κ is a spinor, constant in cylindrical coordinates, that satisfies the projection condition

$$(1 + i\kappa\gamma_{12})\eta_\kappa = 0. \quad (3.18)$$

From (3.6), we therefore see that η_0 is an asymptotically modified-Killing spinor field. A more careful calculation, considering the asymptotic fall-off rates of the metric and matter fields, shows that η_0 actually behaves as

$$\gamma^i \hat{\nabla}_i \eta_0 = \frac{\partial_z A(\theta, z)}{r} \eta_0 + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (3.19)$$

for some function $A(\theta, z)$, defined on the torus at radial infinity.

Now, let us consider the inhomogeneous equation

$$\gamma^i \hat{\nabla}_i \eta_1 = -\gamma^i \hat{\nabla}_i \eta_0, \quad (3.20)$$

subject to the boundary condition that η_1 vanishes asymptotically. It is straightforward to show that $\gamma^i \hat{\nabla}_i \eta = 0$ has no non-zero asymptotically vanishing solutions as long as the inequality (3.16) is satisfied. Therefore, we can formally write down the solution of (3.20) as

$$\eta_1(x) = \int_S dy G(x, y) \gamma^i \hat{\nabla}_i \left(-\gamma^i \hat{\nabla}_i \eta_0(y) \right), \quad (3.21)$$

where $G(x, y)$ is the Green's function of the positive-definite, hermitian second-order operator $-(i\gamma^i \hat{\nabla}_i)^2$.

If this integral converges, it immediately follows that the spinor $\eta = \eta_0 + \eta_1$ is both asymptotically modified-Killing, with limiting value η_0 , and also satisfies the modified Witten-Nester condition throughout S .

To check the convergence of this integral, we perform a Fourier mode expansion of the integrand along the circular z -direction. From (3.19), it is clear that the zero-frequency component of the source term $\gamma^i \hat{\nabla}_i \eta_0$ vanishes as $1/r^2$, whilst all higher frequency components vanish as $1/r$. On the other hand, the zero-frequency component of $G(x, y)$ grows logarithmically at large distances, whilst other frequency components decay exponentially. Putting these results together, we find that this integral is convergent, and that η_1 asymptotically vanishes, at least as fast as $(\log r)/r$.

Bogomol'nyi bounds for the gravitational abelian-Higgs model. In order to verify the inequality (3.16), we need to identify a suitable current \mathcal{J}_μ , which satisfies (3.4). Clearly, the choice of a current \mathcal{J}_μ that satisfies these conditions must be made on a model-by-model basis, as this inequality depends on the form of the energy-momentum tensor.

In fact, for non-gravitational models, a similar inequality,

$$T^0_0 - \kappa(\partial_r J_\theta - \partial_\theta J_r) \geq 0, \quad (3.22)$$

is instrumental in establishing Bogomol'nyi bounds. Therefore, for a given gravitational theory, it would be reasonable to identify \mathcal{J}_μ with the current J_μ that is involved in establishing the Bogomol'nyi bound for the corresponding non-gravitational theory. Having made this guess, we would then need to show that this current both satisfies the condition (3.4) and enables us to establish the inequality (3.16).

We shall begin by examining the gravitational abelian-Higgs model

$$\mathcal{L} = \frac{1}{2}R + \hat{\partial}_\mu\phi^*\hat{\partial}^\mu\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \beta^2(\phi^*\phi - \xi)^2, \quad (3.23)$$

where ϕ is a U(1)-charged scalar field with covariant derivative

$$\hat{\partial}_\mu\phi = \partial_\mu\phi - igA_\mu\phi, \quad (3.24)$$

g is the gauge coupling constant and ξ is a positive constant.

The non-gravitational limit of this theory, the abelian-Higgs model, is the prototype for the traditional Bogomol'nyi procedure: it is the starting point for the Bogomol'nyi rearrangement of the energy-momentum tensors of other non-gravitational theories. Similarly, we shall see that the Bogomol'nyi rearrangement of the gravitational abelian-Higgs theory will enable us to obtain Bogomol'nyi bounds for a variety of gravitational theories.

In the (non-gravitational) abelian-Higgs model, the current

$$J_\mu = \frac{i}{2}[\phi(\hat{\partial}_\mu\phi)^* - \phi^*(\hat{\partial}_\mu\phi)] + g\xi A_\mu, \quad (3.25)$$

satisfies the inequality (3.22) and therefore establishes a Bogomol'nyi bound. J_μ also provides us with the topological charge $Q = n\xi$, due to the boundary conditions satisfied by finite- μ field configurations. Following our earlier discussion, we therefore make the identification $\mathcal{J}_\mu = J_\mu$. It is straightforward to check that the same current produces the conserved topological charge $Q = n\xi$ in the gravitational abelian-Higgs theory. Therefore, we now turn to proving the inequality (3.16) for this choice of current.

The key to verifying this inequality is to notice that, when the parameters β and g are in the Bogomol'nyi limit ($\beta^2 = g^2/2$), the gravitational abelian-Higgs model is the bosonic limit of an $N = 1$ supergravity theory with D -term symmetry breaking — a model with a single charged chiral superfield, simple Kähler potential, a non-zero Fayet-Iliopoulos constant ξ and a superpotential that is identically zero. Furthermore, our choice of \mathcal{J}_μ coincides with the gravitino U(1) connection A_μ^B .

A gravitational Bogomol'nyi bound has already been established for this theory in [11], where it was noticed that the total energy for this system could be written as the sum of squares of the fermionic supersymmetry transformations. In terms of the formalism described here, this is equivalent to showing that the left-hand side of the inequality (3.16) can be written as a sum of squares of certain spinorial quantities, defined as follows:

$$\delta\chi = -\frac{i}{2}\gamma^\mu(\hat{\partial}_\mu\phi)\eta, \quad (3.26)$$

$$\delta\lambda = -\frac{i}{4}\gamma^{\mu\nu}F_{\mu\nu}\eta - \frac{g}{2}(\phi^*\phi - \xi)\eta. \quad (3.27)$$

As the notation suggests, these quantities are clearly related to the higgsino and gaugino supersymmetry transformations respectively. In fact, each is a linear combination of supersymmetry transformations given by the two Weyl components that are encoded in the Dirac spinor η . Furthermore, with \mathcal{J}_μ defined as in (3.25), $\hat{\nabla}_\mu\eta$ is a linear combination of gravitino supersymmetry transformations in the same manner.

With a little effort, one can show that

$$4\overline{\delta\chi}\gamma^\mu\delta\chi = \left[\hat{\partial}^\mu\phi^*\hat{\partial}_\nu\phi + \hat{\partial}_\nu\phi^*\hat{\partial}^\mu\phi - \delta^\mu{}_\nu\hat{\partial}^\rho\phi^*\hat{\partial}_\rho\phi \right] u^\nu - \epsilon^{\mu\nu\rho\sigma}\partial_\nu\mathcal{J}_\rho v_\sigma - \frac{g}{2}(\phi^*\phi - \xi)\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}v_\sigma, \quad (3.28)$$

and

$$2\overline{\delta\lambda}\gamma^\mu\delta\lambda = \left[F^{\mu\rho}F_{\rho\nu} - \delta^\mu{}_\nu\left(-\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma} - \frac{g^2}{2}(\phi^*\phi - \xi)\right) \right] u^\nu + \frac{g}{2}(\phi^*\phi - \xi)\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}v_\sigma. \quad (3.29)$$

Hence we can rewrite the left-hand side of (3.16) as a sum of squares:

$$T^0{}_\nu u^\nu - \epsilon^{0ijk}\partial_i\mathcal{J}_j v_k = 4\delta\chi^\dagger\delta\chi + 2\delta\lambda^\dagger\delta\lambda + \left(\beta^2 - \frac{g^2}{2}\right)(\phi^*\phi - \xi)^2. \quad (3.30)$$

Therefore, provided that $\beta^2 \geq g^2/2$, we obtain the Bogomol'nyi bound

$$\mu \geq 2\pi|n|\xi. \quad (3.31)$$

In the Bogomol'nyi limit $\beta^2 = g^2/2$, this bound is saturated when each positive-definite term in (3.14) vanishes throughout S . This yields the following Bogomol'nyi equations:

$$\delta\chi = \delta\lambda = \hat{\nabla}_i\eta = 0. \quad (3.32)$$

Notice that the Bogomol'nyi equation for η implies that u^μ and v^μ are constant (and Killing) throughout S . The existence of these two Killing vectors implies that minimum-energy cosmic string solutions are static and translationally invariant along the z -axis (cf. the traditional gravitational Bogomol'nyi method, following [2], where these symmetries were assumed, rather than derived). Furthermore, regarding this theory as a D -term supergravity model, solutions of the Bogomol'nyi equations (3.32) partially preserve supersymmetry — i.e. they are BPS solutions.

Bogomol'nyi bounds for other gravitational field theories. Having obtained a Bogomol'nyi bound and Bogomol'nyi equations for the gravitational abelian-Higgs theory, it is now possible to construct Bogomol'nyi bounds and equations for other gravitational field theories.

This is achieved by noticing that any symmetry-breaking term of the form

$$\beta^2[\mathcal{M} - \xi]^2, \quad (3.33)$$

where \mathcal{M} is a real-valued quadratic form with respect to the scalar fields ϕ_i and their complex conjugates ϕ_i^* , can be brought to the form

$$\tilde{\beta}^2\left(\sum_i q_i|\psi_i|^2 - \tilde{\xi}\right)^2, \quad (3.34)$$

where ψ_i are a suitably chosen (charge-preserving) unitary transformation of the fields ϕ_i , and q_i are the charges of the fields ψ_i . Furthermore, such a transformation will leave the kinetic terms in the energy-momentum tensor unchanged:

$$\sum_i |\partial\phi_i|^2 = \sum_i |\partial\psi_i|^2 . \quad (3.35)$$

Therefore, this field transformation effectively turns any theory that contains a symmetry breaking potential of the form (3.33) into an abelian-Higgs theory (perhaps with some extra terms in the scalar potential). Hence we can minimise the energy by applying the abelian-Higgs Bogomol'nyi rearrangement to the energy-momentum tensor, written in terms of the new fields ψ_i .

As a concrete example, let us consider a popular model from $N = 1$ supergravity – an F -term symmetry-breaking model containing three chiral superfields Φ_0 and Φ_{\pm} , with charges 0 and ± 1 , and the superpotential

$$W = \beta\Phi_0\left(\Phi_+\Phi_- - \frac{\xi_F}{2}\right), \quad (3.36)$$

where ξ_F is a positive constant. This superpotential gives rise to the following symmetry-breaking scalar potential:

$$V = \alpha^2(\text{Re } F)^2 + V_{\text{rest}}, \quad (3.37)$$

where

$$\alpha^2 = \beta^2[(1 - |\phi_0|^2)^2 + |\phi_0|^2]e^{\sum_i |\phi_i|^2}, \quad (3.38)$$

$$F = \phi_+\phi_- - \xi_F/2, \quad (3.39)$$

and

$$V_{\text{rest}} = \beta^2|\phi_0|^2e^{\sum_i |\phi_i|^2} \left[|\phi_- + \phi_+^*F|^2 + |\phi_+ + \phi_-^*F|^2 \right] + \alpha^2(\text{Im } F)^2 + \frac{g^2}{2f} \left(|\phi_+|^2 - |\phi_-|^2 \right)^2 . \quad (3.40)$$

The function f is the gauge kinetic function, which appears in the bosonic Lagrangian as follows:

$$\mathcal{L} = \frac{1}{2}R + \sum_i \hat{\partial}_\mu\phi_i^* \hat{\partial}^\mu\phi_i - \frac{f}{4}F^{\mu\nu}F_{\mu\nu} - V . \quad (3.41)$$

We will leave f unspecified for now, so that we can examine how our choice of f affects the existence, and attainability, of a Bogomol'nyi bound for this model.

We shall now consider topological cosmic strings in the bosonic limit of this model. The scalar potential V is manifestly non-negative, and takes the minimum value of zero when $\phi_0 = 0$, $|\phi_+| = \sqrt{\xi}$ and $\phi_- = \phi_+^*$. Therefore the vacuum manifold has a $U(1)$ topology, and this model admits topological cosmic string configurations.

If we now rotate to the fields ψ_0 and ψ_{\pm} , where

$$\psi_0 = \phi_0 \quad \text{and} \quad \psi_{\pm} = \frac{1}{\sqrt{2}}(\phi_{\pm} \pm \phi_{\mp}^*), \quad (3.42)$$

then we find that V becomes

$$V = \frac{\alpha^2}{4} \left(|\psi_+|^2 - |\psi_-|^2 - \xi_F \right)^2 + V_{\text{rest}}, \quad (3.43)$$

whilst the kinetic terms in the energy-momentum tensor are unchanged. Therefore, written in terms of the new fields ψ_0 and ψ_{\pm} , this F -term model has an energy-momentum tensor which is the sum of the abelian-Higgs energy-momentum tensor and the scalar potential V_{rest} .

Hence, by generalising $\delta\chi$ and $\delta\lambda$ as follows

$$\delta\chi_i = -\frac{i}{2} \gamma^\mu (\hat{\partial}_\mu \psi_i) \eta, \quad (3.44)$$

$$\delta\lambda = -\frac{i}{4} \gamma^{\mu\nu} F_{\mu\nu} \eta - \frac{g}{2} \left(\sum_i q_i |\psi_i|^2 - \xi_F \right) \eta, \quad (3.45)$$

and using the current

$$\mathcal{J}_\mu = \frac{i}{2} \sum_i q_i [\psi_i (\hat{\partial}_\mu \psi_i)^* - \psi_i^* (\hat{\partial}_\mu \psi_i)] + g \xi_F A_\mu, \quad (3.46)$$

which can easily be shown to satisfy the condition (3.4), thereby enabling the existence of asymptotically modified-Killing spinors, we find that

$$T^0_\nu u^\nu - \epsilon^{0ijk} \partial_i \mathcal{J}_j v_k = \sum_i 4\delta\chi_i^\dagger \delta\chi_i + 2\delta\lambda^\dagger \delta\lambda + \frac{1}{4f} (\alpha^2 f - 2g^2) \left(\sum_i q_i |\psi_i|^2 - \xi_F \right)^2 + V_{\text{rest}}. \quad (3.47)$$

Substituting this into (3.14), we obtain the Bogomol'nyi bound

$$\mu \geq 2\pi |n| \xi_F, \quad (3.48)$$

provided that f satisfies the inequality $\alpha^2 f \geq 2g^2$.

For the usual choice $f = 1$, this inequality is satisfied as long as $\beta^2 = 2g^2$, since $\alpha^2 \geq 1$ everywhere. However, the inequality cannot be saturated everywhere, and hence this Bogomol'nyi bound cannot be saturated either.

For this Bogomol'nyi bound to be attainable, we need to choose f according to the formula $\alpha^2 f = 2g^2$. This corresponds to the Bogomol'nyi limit, or critical coupling, in the abelian-Higgs model, where we had to relate the gauge coupling constant to the mass of the scalar field in order to obtain an attainable bound. For this choice of f , this Bogomol'nyi bound is saturated by field configurations that satisfy the following Bogomol'nyi equations:

$$\delta\chi_i = \delta\lambda = \hat{\nabla}_i \eta = V_{\text{rest}} = 0. \quad (3.49)$$

We note that, just as the non-gravitational version of this model has embedded Nielsen-Olesen strings as minimum-energy solutions, the above Bogomol'nyi equations are solved by embedded minimum-energy solutions of the gravitational abelian-Higgs model.

Let us now consider the relationship between these results and supersymmetry. The Lagrangian we have just considered is derived from the bosonic limit of an F -term $N = 1$

supergravity model. For this model, with its non-vanishing superpotential, it is easily seen that the BPS equations cannot be satisfied [14] — i.e. there are no non-trivial minimum-energy configurations that preserve any degree of supersymmetry.³ Therefore, it may initially seem rather surprising that we have been able to obtain an energetic Bogomol’nyi bound for this Lagrangian. However, on closer inspection, this result turns out to be a consistent generalisation of analogous results for F -term strings in both (global) supersymmetry and supergravity in the cylindrically symmetric limit — that, although the BPS equations cannot be satisfied, one can still establish an energetic Bogomol’nyi bound [4].

The relationship between our energetic Bogomol’nyi bound and the non-existence of supersymmetric solutions manifests itself in a number of ways. Firstly, it is clear that the Bogomol’nyi equations (3.49) are not equivalent to the F -term BPS equations. Secondly, the Bogomol’nyi bound is only attainable when $\alpha^2 f = 2g^2$ — a choice which would result in f not being holomorphic, and therefore not a valid choice if (3.41) is to be the bosonic part of a supergravity Lagrangian. Furthermore, and most importantly, the current \mathcal{J}_μ is not the gravitino $U(1)$ connection A_μ^B — a key result that enabled us to find an appropriate spinor parameter η for the Witten-Nester energy when Killing spinors, in the usual supergravity sense, cannot exist for $\delta > 0$.

4. Comparison to Bogomol’nyi bounds for non-gravitational theories

There is a strong analogy between the gravitational Bogomol’nyi method we have presented here and the traditional Bogomol’nyi procedure for vortices in non-gravitational models. In fact, it is more appropriate to say that our new method is really a generalisation of the non-gravitational Bogomol’nyi method.

This analogy begins with the Witten-Nester energy expression which, as mentioned earlier, is the gravitational energy expression that is closest, for our purposes, to the definition of the total energy in Minkowski spacetime. Using the Witten-Nester energy, we were able to come up with an inequality (3.16), that must be satisfied in order to establish a gravitational Bogomol’nyi bound. As we saw there, this inequality is a generalised version of the inequality (3.22) that is obtained by the non-gravitational Bogomol’nyi procedure.

Furthermore, the identities (3.28) and (3.29) are generalisations of the Minkowski spacetime identities

$$\left| \hat{\partial}_r \phi \pm \frac{i}{r} \hat{\partial}_\theta \phi \right|^2 = |\hat{\partial}_r \phi|^2 + \frac{1}{r^2} |\hat{\partial}_\theta \phi|^2 + \mp \frac{1}{r} (\partial_r \mathcal{J}_\theta - \partial_\theta \mathcal{J}_r) \mp \frac{g}{r} F_{r\theta} (\phi^* \phi - \xi), \quad (4.1)$$

and

$$\frac{1}{2} \left(\frac{F_{r\theta}}{r} \pm g(\phi^* \phi - \xi) \right)^2 = \frac{F_{r\theta}^2}{2r^2} + \frac{g^2}{2} (\phi^* \phi - \xi)^2 \pm \frac{g}{r} F_{r\theta} (\phi^* \phi - \xi), \quad (4.2)$$

that are used in the non-gravitational Bogomol’nyi method to write the left-hand side of (3.22) as a sum of squares.

³In this paper, we take Bogomol’nyi equations to be the equations that characterise minimum-energy solutions, whilst BPS equations are the equations that characterise partially supersymmetric field configurations. Although these two properties often come hand in hand, this is not the case here, and we must therefore distinguish between the equations that characterise these two properties.

Finally, the field transformation technique discussed above — which relates many gravitational field theories to the gravitational abelian-Higgs model, and therefore allows us to construct Bogomol’nyi bounds for these theories — also relates the Bogomol’nyi bounds and equations of the non-gravitational versions of these theories to the non-gravitational abelian-Higgs model.

This correspondence between the gravitational Bogomol’nyi method presented here and the traditional Bogomol’nyi method for non-gravitational theories demonstrates that non-gravitational Bogomol’nyi bounds do survive coupling to gravity. Wherever we can construct a Bogomol’nyi bound for a non-gravitational theory, we can use the techniques described above to generalise this Bogomol’nyi rearrangement, and therefore provide a Bogomol’nyi bound for the gravitational version of the same theory.

In this manner, we can establish Bogomol’nyi bounds for many other gravitational theories that are currently of cosmological interest — such as P -term models [5], semi-local models [15], and D -term models with non-zero superpotentials — where Bogomol’nyi rearrangements are already known in the non-gravitational limit.

5. Conclusion

We have presented a general method for establishing Bogomol’nyi bounds and finding minimum-energy cosmic string solutions that can be applied to a wide range of gravitational field theories that contain symmetry-breaking scalar potentials. Unlike the traditional method for establishing energy bounds for gravitational theories [2], this new method does not involve making any prior assumptions about the symmetries of minimum-energy solutions.

Our work generalises the results of [10] and [11], regarding certain D -term supergravity models. Although the algebraic manipulations that enabled us to derive these bounds were borrowed from D -term supergravity, they were actually found to be applicable to a wide variety of (possibly non-supersymmetric) theories.

In fact we have seen, in section 4, that these manipulations, although taken from a supersymmetric context, are really the covariant generalisations of the key identities that were used to derive the non-gravitational Bogomol’nyi bound for the abelian-Higgs model. In this sense, our procedure is really a covariant generalisation of the traditional Bogomol’nyi procedure. Therefore, we can confirm that all results that have been proven so far by making assumptions about the symmetry of the metric and using the traditional Bogomol’nyi technique, still hold when one allows for asymmetric perturbations.

We applied our technique to the particular example of the bosonic Lagrangian for F -term strings in $N = 1$ supergravity. Here, we found an energetic Bogomol’nyi bound and corresponding Bogomol’nyi equations — however, this bound is unattainable for any holomorphic choice of gauge kinetic function. These results, which have been derived for cylindrically symmetric strings in [4], have therefore been generalised by this technique to cover cylindrically asymmetric field configurations.

The Bogomol’nyi equations obtained using this new technique confirm that minimum-energy solutions are static and straight — symmetries that were previously assumed rather

than proved. Furthermore these equations allow for the same cylindrically symmetric single-vortex solutions as the Bogomol'nyi equations of the traditional method, thereby confirming the stability of these solutions against decay to asymmetric field configurations of lower energy.

However, the new Bogomol'nyi equations do not necessarily imply that minimum-energy solutions must be cylindrically symmetric. Therefore, it would be interesting to look for multi-string configurations that saturate the Bogomol'nyi bound, in analogy with the static multi-vortex solutions that exist in non-gravitational field theories. To find such solutions, one would presumably have to repeat Taubes's analysis of the abelian-Higgs Bogomol'nyi equations [16] for the gravitational Bogomol'nyi equations (3.32).

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A. The energy of cosmic string spacetimes

In section 2 we claimed that any surface integral expression for the energy of a canonical asymptotically flat spacetime can also be used to calculate the energy of a cosmic string spacetime, provided that the surface of integration is changed from the sphere at spatial infinity to the torus at radial infinity. Then, in section 3, we employed this result to interpret the surface integral of $E^{\mu\nu}$.

There are many ways to check this assertion. One would be to see whether an energy expression obtained in this manner is equivalent to the gravitational Hamiltonian obtained by following the background-subtraction procedure described by Hawking and Horowitz [17]. However, here we shall adopt a more pedagogical approach by employing the principles that were used to define the ADM energy of canonical asymptotically flat spacetimes to define an equivalent energy for cosmic string spacetimes.

We begin by considering linearised gravity on a static, cylindrically symmetric background. More specifically, let us consider a spacetime that admits global cylindrical polar coordinates, in which the metric may be written

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{A.1}$$

where

$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu = dt^2 - dr^2 - C(r)^2 d\theta^2 - dz^2 \tag{A.2}$$

is a static, cylindrically symmetric background metric, and ϵ is a small parameter, with respect to which we shall linearise all quantities. We shall also assume that $C'(0) = 0$ and $C(r) \rightarrow (1 - \frac{\delta}{2\pi})r$ as $r \rightarrow \infty$, so that the background metric is completely regular, with deficit angle δ .

In the linearised theory, the dynamics of the matter fields take place with respect to the fixed background (to leading order), and therefore the energy may be defined in a

special-relativistic sense, in terms of the linearised energy-momentum tensor:

$$p^\mu u_\mu = \int_S d^3x \sqrt{-\tilde{g}} T^{(L)0}{}_\nu u^\nu, \tag{A.3}$$

where S is a background-static, spacelike hypersurface whose normal vector points in the (timelike and background-Killing) t -direction, and u^μ is a constant, timelike vector that represents the 4-velocity of the observer at spacelike infinity who is measuring the energy of the system. Due to Einstein's equation, we can replace $T^{(L)\mu}{}_\nu$ with $G^{(L)\mu}{}_\nu$, the linearised Einstein tensor, in the above expression.

It is possible to express the linearised Riemann tensor $R^{(L)\mu\nu}{}_{\rho\sigma}$ in terms of the background Riemann tensor $\tilde{R}^{\mu\nu}{}_{\rho\sigma}$, the background metric connection $\tilde{\Gamma}^{\mu\nu}{}_\rho$, and the linearised perturbation of the spin connection $\Delta\omega_\rho{}^{\mu\nu}$, in the following manner:

$$R^{(L)\mu\nu}{}_{\rho\sigma} = \tilde{R}^{\mu\nu}{}_{\rho\sigma} + 2 \left(\Delta\omega_{[\rho}{}^{\mu\nu}{}_{\sigma]} - 2\Delta\omega_{[\rho}{}^{\tau[\mu}\tilde{\Gamma}^{\nu]}{}_{\sigma]\tau} \right). \tag{A.4}$$

With the aid of this expression, along with the identity

$$G^\mu{}_\nu = \frac{1}{4} \epsilon^{\rho\mu\gamma\delta} \epsilon_{\rho\nu\alpha\beta} R^{\alpha\beta}{}_{\gamma\delta}, \tag{A.5}$$

we can rewrite (A.3) as follows:

$$p^\mu u_\mu = 2\pi\delta L_z + \frac{1}{4} \int_{\partial S} dS_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\delta\alpha\beta\sigma} \Delta\omega_\rho{}^{\alpha\beta} u^\delta. \tag{A.6}$$

We note that this expression is formally equivalent to Nester's expression of the ADM energy in [13].

Now that we have an expression for the energy which only depends on the *asymptotic* behaviour of the system, we may allow u^μ to be any *asymptotically* constant, timelike vector field. Similarly, we may now allow S to be any maximal spacelike hypersurface, with a timelike, *asymptotically* Killing normal vector. In this manner, we obtain an expression for the energy which only depends on the asymptotic form of the metric.

Now we invoke the canonical argument that, on physical grounds, we require the mass of a system to be determined purely by the long-distance behaviour of the metric, and hence be independent of the fields in the interior of the system. This implies that the expression (A.6) represents the total energy of *any* spacetime that is asymptotically cylindrically symmetric, irrespective of its behaviour in the interior. In other words, (A.6) represents the energy of any cosmic string spacetime.

It is straightforward to check that the surface integral of $E^{\mu\nu}$ equals this energy expression, by decomposing the spin connection in the asymptotic region as

$$\omega_\mu{}^{\alpha\beta} = \tilde{\omega}_\mu{}^{\alpha\beta} + \Delta\omega_\mu{}^{\alpha\beta}, \tag{A.7}$$

where $\tilde{\omega}_\mu{}^{\alpha\beta}$ is the spin connection for a cylindrically symmetric metric of identical conical deficit angle.

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